J Glob Optim (2010) 46:49–62
DOI 10.1007/s10898-009-9408-0

# A non-smooth three critical points theorem with applications in differential inclusions

**Alexandru Kristály · Waclaw Marzantowicz · Csaba Varga**

Received: 22 December 2008 / Accepted: 7 February 2009 / Published online: 22 February 2009


**Abstract** We extend a recent result of Ricceri concerning the existence of three critical points of certain non-smooth functionals. Two applications are given, both in the theory of differential inclusions; the first one concerns a non-homogeneous Neumann boundary value problem, the second one treats a quasilinear elliptic inclusion problem in the whole $\mathbb{R}^N$.

**Keywords** Locally Lipschitz functions · Critical points · Differential inclusions

## 1 Introduction and prerequisites

It is a simple exercise to show that a $C^1$ function $f : \mathbb{R} \to \mathbb{R}$ having two local minima has necessarily a third critical point. However, once we are dealing with functions defined on a multi-dimensional space, the problem becomes much deeper. Motivated mostly by various real-life phenomena coming from Mechanics and Mathematical Physics, the latter problem has been treated by several authors, see Pucci-Serrin [13], Ricceri [14–17], Marano-Motreanu [10], Arcoya-Carmona [1], Bonanno [3,2], Bonanno-Candito [4].

The aim of the present paper is to give an extension of the very recent three critical points theorem of Ricceri [17] to locally Lipschitz functions, providing also two applications in partial differential inclusions; the first one for a non-homogeneous Neumann boundary value problem, the second one for a quasilinear elliptic inclusion problem in $\mathbb{R}^N$.

A. Kristály (✉)
Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania
e-mail: alexandrukristaly@yahoo.com

W. Marzantowicz
Department of Mathematics, Adam Mickiewicz University, Poznan, Poland

C. Varga
Department of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

In order to do that, we recall two results which are crucial in our further investigations. The first result is due to Ricceri [18] guaranteeing the existence of two local minima for a parametric functional defined on a Banach space. Note that no smoothness assumption is required on the functional.

**Theorem 1.1** ([18], Theorem 4) *Let $X$ be a real, reflexive Banach space, let $\Lambda \subseteq \mathbb{R}$ be an interval, and let $\varphi : X \times \Lambda \to \mathbb{R}$ be a function satisfying the following conditions:*

1. *$\varphi(x, \cdot)$ is concave in $\Lambda$ for all $x \in X$;*
2. *$\varphi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in $X$ for all $\lambda \in \Lambda$;*
3. $\beta_1 := \sup_{\lambda\in\Lambda} \inf_{x\in X} \varphi(x, \lambda) < \inf_{x\in X} \sup_{\lambda\in\Lambda} \varphi(x, \lambda) =: \beta_2$.

*Then, for each $\sigma > \beta_1$ there exists a non-empty open set $\Lambda_0 \subset \Lambda$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly lower semicontinuous function $\Phi : X \to \mathbb{R}$, there exists $\mu_0 > 0$ such that, for each $\mu \in ]0, \mu_0[$, the function $\varphi(\cdot, \lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X : \varphi(x, \lambda) < \sigma\}$.*

The second main tool in our argument is the "zero-altitude" Mountain Pass Theorem for locally Lipschitz functionals, due to Motreanu-Varga [12]. Before giving this result, we are going to recall some basic properties of the generalized directional derivative as well as of the generalized gradient of a locally Lipschitz functional which will be used later.

Let $(X, \|\cdot\|)$ be a Banach space.

**Definition 1.1** A function $\Phi : X \to \mathbb{R}$ is locally Lipschitz if, for every $x \in X$, there exist a neighborhood $U$ of $x$ and a constant $L > 0$ such that

$$|\Phi(y) - \Phi(z)| \leq L\|y - z\| \quad \text{for all } y, z \in U.$$

Although it is not necessarily differentiable in the classical sense, a locally Lipschitz function admits a derivative, defined as follows:

**Definition 1.2** The generalized directional derivative of $\Phi$ at the point $x \in X$ in the direction $y \in X$ is

$$\Phi^\circ(x; y) = \limsup_{z\to x,\ \tau\to 0^+} \frac{\Phi(z + \tau y) - \Phi(z)}{\tau}.$$

The generalized gradient of $\Phi$ at $x \in X$ is the set

$$\partial\Phi(x) = \{x^\star \in X^\star : \langle x^\star, y\rangle \leq \Phi^\circ(x; y) \quad \text{for all } y \in X\}.$$

For all $x \in X$, the functional $\Phi^\circ(x, \cdot)$ is subadditive and positively homogeneous; thus, due to the Hahn–Banach theorem, the set $\partial\Phi(x)$ is nonempty. In the sequel, we resume the main properties of the generalized directional derivatives.

**Lemma 1.1** [7] *Let $\Phi, \Psi : X \to \mathbb{R}$ be locally Lipschitz functions. Then,*

(a) $\Phi^\circ(x; y) = \max\{\langle\xi, y\rangle : \xi \in \partial\Phi(x)\}$;
(b) $(\Phi + \Psi)^\circ(x; y) \leq \Phi^\circ(x; y) + \Psi^\circ(x; y)$;
(c) $(-\Phi)^\circ(x; y) = \Phi^\circ(x; -y)$; *and* $\Phi^\circ(x; \lambda y) = \lambda\Phi^\circ(x; y)$ *for every* $\lambda > 0$;
(d) *The function $(x, y) \mapsto \Phi^\circ(x; y)$ is upper semicontinuous.*

The next definition generalizes the notion of critical point to the non-smooth context:

**Definition 1.3** [6] A point $x \in X$ is a critical point of $\Phi : X \to \mathbb{R}$, if $0 \in \partial\Phi(x)$, that is,

$$\Phi^\circ(x; y) \geq 0 \quad \text{for all } y \in X.$$

For every $c \in \mathbb{R}$, we denote by $K_c = \{x \in X : 0 \in \partial\Phi(x),\ \Phi(x) = c\}$.

*Remark 1.1* Note that every local extremum point of the locally Lipschitz function $\Phi$ is a critical point of $\Phi$ in the sense of Definition 1.3.

**Definition 1.4** The locally Lipschitz function $\Phi : X \to \mathbb{R}$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition), if every sequence $\{x_n\}$ in $X$ such that

$(PS_1)$ $\Phi(x_n) \to c$ as $n \to \infty$;
$(PS_2)$ there exists a sequence $\{\varepsilon_n\}$ in $]0, +\infty[$ with $\varepsilon_n \to 0$ such that $\Phi^\circ(x_n; y - x_n) + \varepsilon_n \|y - x_n\| \geq 0$ for all $y \in X, n \in \mathbb{N}$,

admits a convergent subsequence.

We recall now the zero-altitude version of the Mountain Pass Theorem, due to Motreanu-Varga [12].

**Theorem 1.2** *Let $E : X \to \mathbb{R}$ be a locally Lipschitz function satisfying $(PS)_c$ for all $c \in \mathbb{R}$. If there exist $x_1, x_2 \in X$, $x_1 \neq x_2$ and $r \in (0, \|x_2 - x_1\|)$ such that*

$$\inf\{E(x) : ||x - x_1|| = r\} \geq \max\{E(x_1), E(x_2)\},$$

*and we denote by $\Gamma$ the family of continuous paths $\gamma : [0, 1] \to X$ joining $x_1$ and $x_2$, then*

$$c := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} E(\gamma(s)) \geq \max\{E(x_1), E(x_2)\}$$

*is a critical value for $E$ and $K_c \setminus \{x_1, x_2\} \neq \emptyset$.*

## 2 Main result: non-smooth Ricceri's multiplicity theorem

For every $\tau \geq 0$, we introduce the following class of functions:
$(\mathcal{G}_\tau)$ : $g \in C^1(\mathbb{R}, \mathbb{R})$ is bounded, and $g(t) = t$ for any $t \in [-\tau, \tau]$.
The main result of this paper is the following.

**Theorem 2.1** *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $\tilde{X}_i$ $(i = 1, 2)$ be two Banach spaces such that the embeddings $X \hookrightarrow \tilde{X}_i$ are compact. Let $\Lambda$ be a real interval, $h : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function, and let $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ be two locally Lipschitz functions such that $E_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g \circ \Phi_2$ restricted to $X$ satisfies the $(PS)_c$-condition for every $c \in \mathbb{R}$, $\lambda \in \Lambda$, $\mu \in [0, |\lambda| + 1]$ and $g \in \mathcal{G}_\tau$, $\tau \geq 0$. Assume that $h(\|\cdot\|) + \lambda\Phi_1$ is coercive on $X$ for all $\lambda \in \Lambda$ and that there exists $\rho \in \mathbb{R}$ such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)]. \qquad (2.1)$$

*Then, there exist a non-empty open set $A \subset \Lambda$ and $r > 0$ with the property that for every $\lambda \in A$ there exists $\mu_0 \in ]0, |\lambda| + 1]$ such that, for each $\mu \in [0, \mu_0]$ the functional $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$ has at least three critical points in $X$ whose norms are less than $r$.*

*Proof* Since $h$ is a non-decreasing convex function, $X \ni x \mapsto h(\|x\|)$ is also convex; thus, $h(\|\cdot\|)$ is sequentially weakly lower semicontinuous on $X$, see Brézis [5, Corollaire III.8]. From the fact that the embeddings $X \hookrightarrow \tilde{X}_i$ $(i = 1, 2)$ are compact and $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ are locally Lipschitz functions, it follows that the function $E_{\lambda,\mu}$ as well as $\varphi : X \times \Lambda \to \mathbb{R}$ (in the first variable) given by

$$\varphi(x, \lambda) = h(\|x\|) + \lambda(\Phi_1(x) + \rho)$$

are sequentially weakly lower semicontinuous on $X$.

The function $\varphi$ satisfies the hypotheses of Theorem 1.1. Fix $\sigma > \sup_\Lambda \inf_X \varphi$ and consider a nonempty open set $\Lambda_0$ with the property expressed in Theorem 1.1. Let $A = [a, b] \subset \Lambda_0$.

Fix $\lambda \in [a, b]$; then, for every $\tau \geq 0$ and $g_\tau \in \mathcal{G}_\tau$, there exists $\mu_\tau > 0$ such that, for any $\mu \in ]0, \mu_\tau[$, the functional $E^\tau_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g_\tau \circ \Phi_2$ restricted to $X$ has two local minima, say $x_1^\tau, x_2^\tau$, lying in the set $\{x \in X : \varphi(x, \lambda) < \sigma\}$.

Note that

$$\bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \subset \{x \in X : h(\|x\|) + a\Phi_1(x) < \sigma - a\rho\}$$
$$\cup \{x \in X : h(\|x\|) + b\Phi_1(x) < \sigma - b\rho\}.$$

Because the function $h(\|\cdot\|) + \lambda\Phi_1$ is coercive on $X$, the set on the right-side is bounded. Consequently, there is some $\eta > 0$, such that

$$\bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \subset B_\eta, \tag{2.2}$$

where $B_\eta = \{x \in X : \|x\| < \eta\}$. Therefore,

$$x_1^\tau, x_2^\tau \in B_\eta.$$

Now, set $c^\star = \sup_{t\in[0,\eta]} h(t) + \max\{|a|, |b|\} \sup_{B_\eta} |\Phi_1|$ and fix $r > \eta$ large enough such that for any $\lambda \in [a, b]$ to have

$$\{x \in X : h(\|x\|) + \lambda\Phi_1(x) \leq c^\star + 2\} \subset B_r. \tag{2.3}$$

Let $r^\star = \sup_{B_r} |\Phi_2|$ and correspondingly, fix a function $g = g_{r^*} \in \mathcal{G}_{r^*}$. Let us define $\mu_0 = \min\left\{|\lambda| + 1, \frac{1}{1+\sup|g|}\right\}$. Since the functional $E_{\lambda,\mu} = E^{r^*}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g_{r^*} \circ \Phi_2$ restricted to $X$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, $\mu \in [0, \mu_0]$, and $x_1 = x_1^{r^*}, x_2 = x_2^{r^*}$ are local minima of $E_{\lambda,\mu}$, we may apply Theorem 1.2, obtaining that

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} E_{\lambda,\mu}(\gamma(s)) \geq \max\{E_{\lambda,\mu}(x_1), E_{\lambda,\mu}(x_2)\} \tag{2.4}$$

is a critical value for $E_{\lambda,\mu}$, where $\Gamma$ is the family of continuous paths $\gamma : [0, 1] \to X$ joining $x_1$ and $x_2$. Therefore, there exists $x_3 \in X$ such that

$$c_{\lambda,\mu} = E_{\lambda,\mu}(x_3) \quad \text{and} \quad 0 \in \partial E_{\lambda,\mu}(x_3).$$

If we consider the path $\gamma \in \Gamma$ given by $\gamma(s) = x_1 + s(x_2 - x_1) \subset B_\eta$ we have

$$\begin{aligned} h(\|x_3\|) + \lambda\Phi_1(x_3) &= E_{\lambda,\mu}(x_3) - \mu g(\Phi_2(x_3)) \\ &= c_{\lambda,\mu} - \mu g(\Phi_2(x_3)) \\ &\leq \sup_{s\in[0,1]} (h(\|\gamma(s)\|) + \lambda\Phi_1(\gamma(s)) + \mu g(\Phi_2(\gamma(s)))) - \mu g(\Phi_2(x_3)) \\ &\leq \sup_{t\in[0,\eta]} h(t) + \max\{|a|, |b|\} \sup_{B_\eta} |\Phi_1| + 2\mu_0 \sup |g| \\ &\leq c^\star + 2. \end{aligned}$$

From (2.3) it follows that $x_3 \in B_r$. Therefore, $x_i$, $i = 1, 2, 3$ are critical points for $E_{\lambda,\mu}$, all belonging to the ball $B_r$. It remains to prove that these elements are critical points not only for $E_{\lambda,\mu}$ but also for $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$. Let $x = x_i$, $i \in \{1, 2, 3\}$. Since $x \in B_r$, we have that $|\Phi_2(x)| \leq r^*$. Note that $g(t) = t$ on $[-r^*, r^*]$; thus, $g(\Phi_2(x)) = \Phi_2(x)$. Consequently, on the open set $B_r$ the functionals $E_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,\mu}$ coincide, which completes the proof. □

## 3 Applications

### 3.1 A differential inclusion with non-homogeneous boundary condition

Let $\Omega$ be a non-empty, bounded, open subset of the real Euclidian space $\mathbb{R}^N$, $N \geq 3$, having a smooth boundary $\partial\Omega$ and let $W^{1,2}(\Omega)$ be the closure of $C^\infty(\Omega)$ with the respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 + \int_\Omega u^2(x)\right)^{1/2}.$$

Denote by $2^\star = \frac{2N}{N-2}$ and $\overline{2}^\star = \frac{2(N-1)}{N-2}$ the critical Sobolev exponent for the embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ and for the trace mapping $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega)$, respectively. If $p \in [1, 2^\star]$ then the embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous while if $p \in [1, 2^\star[$, it is compact. In the same way for $q \in [1, \overline{2}^\star]$, $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is continuous, and for $q \in [1, \overline{2}^\star[$ it is compact. Therefore, there exist constants $c_p, \overline{c}_q > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq c_p\|u\|, \quad \text{and} \quad \|u\|_{L^q(\partial\Omega)} \leq \overline{c}_q\|u\|, \quad \forall u \in W^{1,2}(\Omega).$$

Now, we consider a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions:

**(F0)** $F(0) = 0$ and there exists $C_1 > 0$ and $p \in [1, 2^\star[$ such that

$$|\xi| \leq C_1(1 + |t|^{p-1}), \quad \forall \xi \in \partial F(t), \quad t \in \mathbb{R}; \tag{3.1}$$

**(F1)** $\lim_{t\to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0$;

**(F2)** $\limsup_{|t|\to+\infty} \frac{F(t)}{t^2} \leq 0$;

**(F3)** There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$.

*Example 3.1* Let $p \in ]1, 2]$ and $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(t) = \min\{|t|^{p+1}, \arctan(t_+)\}$, where $t_+ = \max\{t, 0\}$. The function $F$ enjoys properties **(F0–F3)**.

Let also $G : \mathbb{R} \to \mathbb{R}$ be another locally Lipschitz function satisfying the following condition:

**(G)** There exists $C_2 > 0$ and $q \in [1, \overline{2}^{\star}[$ such that

$$|\xi| \leq C_2(1 + |t|^{q-1}), \quad \forall \xi \in \partial G(t), \quad t \in \mathbb{R}. \tag{3.2}$$

For $\lambda, \mu > 0$, we consider the following differential inclusion problem, with inhomogeneous Neumann condition:

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta u + u \in \lambda \partial F(u(x)) & \text{in } \Omega; \\ \dfrac{\partial u}{\partial n} \in \mu \partial G(u(x)) & \text{on } \partial\Omega. \end{cases}$$

**Definition 3.1** We say that $u \in W^{1,2}(\Omega)$ is a solution of the problem $(P_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for a.e. $x \in \Omega$ such that for all $v \in W^{1,2}(\Omega)$ we have

$$\int_\Omega (-\Delta u + u) v \mathrm{d}x = \lambda \int_\Omega \xi_F v \mathrm{d}x \quad \text{and} \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} v \mathrm{d}\sigma = \mu \int_{\partial\Omega} \xi_G v d\sigma.$$

The main result of this section reads as follows.

**Theorem 3.1** *Let $F, G : \mathbb{R} \to \mathbb{R}$ be two locally Lipschitz functions satisfying the conditions* **(F0–F3)** *and* **(G)**. *Then there exists a non-degenerate compact interval $[a, b] \subset ]0, +\infty[$ and a number $r > 0$, such that for every $\lambda \in [a, b]$ there exists $\mu_0 \in ]0, \lambda + 1]$ such that for each $\mu \in [0, \mu_0]$, the problem $(P_{\lambda,\mu})$ has at least three distinct solutions with $W^{1,2}$-norms less than $r$.*

In the sequel, we are going to prove Theorem 3.1, assuming from now one that its assumptions are verified.

Since $F, G$ are locally Lipschitz, it follows trough (3.1) and (3.2) in a standard way that $\Phi_1 : L^p(\Omega) \to \mathbb{R}$ $(p \in [1, 2^\star])$ and $\Phi_2 : L^q(\partial\Omega) \to \mathbb{R}$ $(q \in [1, \overline{2}^\star])$ defined by

$$\Phi_1(u) = -\int_\Omega F(u(x)) \mathrm{d}x \quad (u \in L^p(\Omega)) \quad \text{and} \quad \Phi_2(u) = -\int_{\partial\Omega} G(u(x)) \mathrm{d}\sigma \quad (u \in L^q(\partial\Omega))$$

are well-defined, locally Lipschitz functionals and due to Clarke [7, Theorem 2.7.5], we have

$$\partial\Phi_1(u) \subseteq -\int_\Omega \partial F(u(x)) \mathrm{d}x \quad (u \in L^p(\Omega)), \quad \partial\Phi_2(u) \subseteq -\int_{\partial\Omega} \partial G(u(x)) d\sigma \quad (u \in L^q(\partial\Omega)).$$

We introduce the energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,2}(\Omega) \to \mathbb{R}$ associated to the problem $(P_{\lambda,\mu})$, given by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 + \lambda\Phi_1(u) + \mu\Phi_2(u), \quad u \in W^{1,2}(\Omega).$$

Using the latter inclusions and the Green formula, the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(P_{\lambda,\mu})$ in the sense of Definition 3.1. Before proving Theorem 3.1, we need the following auxiliary result.

**Proposition 3.1** $\lim_{t \to 0^+} \dfrac{\inf\{\Phi_1(u) : u \in W^{1,2}(\Omega), \ \|u\|^2 < 2t\}}{t} = 0.$

*Proof* Fix $\tilde{p} \in ]\max\{2, p\}, 2^\star[$. Applying Lebourg's mean value theorem and using **(F0)** and **(F1)**, for any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that

$$|F(t)| \leq \varepsilon t^2 + K(\varepsilon)|t|^{\tilde{p}} \quad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

Taking into account (3.3) and the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^{\tilde{p}}(\Omega)$ we have

$$\Phi_1(u) \geq -\varepsilon c_2^2 \|u\|^2 - K(\varepsilon) c_{\tilde{p}}^{\tilde{p}} \|u\|^{\tilde{p}}, \quad u \in W^{1,2}(\Omega). \tag{3.4}$$

For $t > 0$ define the set $S_t = \{u \in W^{1,2}(\Omega) : \|u\|^2 < 2t\}$. Using (3.4) we have

$$0 \geq \frac{\inf_{u \in S_t} \Phi_1(u)}{t} \geq -2c_2^2 \varepsilon - 2^{\tilde{p}/2} K(\varepsilon) c_{\tilde{p}}^{\tilde{p}} t^{\frac{\tilde{p}}{2}-1}.$$

Since $\varepsilon > 0$ is arbitrary and since $t \to 0^+$, we get the desired limit. □

*Proof of Theorem 3.1* Let us define the function for every $t > 0$ by

$$\beta(t) = \inf\left\{\Phi_1(u) : u \in W^{1,2}(\Omega),\ \frac{\|u\|^2}{2} < t\right\}.$$

We have that $\beta(t) \leq 0$, for $t > 0$, and Proposition 3.1 yields that

$$\lim_{t \to 0^+} \frac{\beta(t)}{t} = 0. \tag{3.5}$$

We consider the constant function $u_0 \in W^{1,2}(\Omega)$ by $u_0(x) = \tilde{t}$ for every $x \in \Omega$, $\tilde{t}$ being from **(F3)**. Note that $\tilde{t} \neq 0$ (since $F(0) = 0$), so $\Phi_1(u_0) < 0$. Therefore it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < -\Phi_1(u_0)\left[\frac{\|u_0\|^2}{2}\right]^{-1}.$$

By (3.5) we get the existence of a number $t_0 \in \left(0, \frac{\|u_0\|^2}{2}\right)$ such that $-\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) > \left[\frac{\|u_0\|^2}{2}\right]^{-1} \Phi_1(u_0) t_0. \tag{3.6}$$

Due to the choice of $t_0$ and using (3.6), we conclude that there exists $\rho_0 > 0$ such that

$$-\beta(t_0) < \rho_0 < -\Phi_1(u_0)\left[\frac{\|u_0\|^2}{2}\right]^{-1} t_0 < -\Phi_1(u_0). \tag{3.7}$$

Define now the function $\varphi : W^{1,2}(\Omega) \times \mathbb{I} \to \mathbb{R}$ by

$$\varphi(u, \lambda) = \frac{\|u\|^2}{2} + \lambda \Phi_1(u) + \lambda \rho_0,$$

where $\mathbb{I} = [0, +\infty)$. We prove that the function $\varphi$ satisfies the inequality

$$\sup_{\lambda \in \mathbb{I}} \inf_{u \in W^{1,2}(\Omega)} \varphi(u, \lambda) < \inf_{u \in W^{1,2}(\Omega)} \sup_{\lambda \in \mathbb{I}} \varphi(u, \lambda). \tag{3.8}$$

The function

$$\mathbb{I} \ni \lambda \mapsto \inf_{u \in W^{1,2}(\Omega)} \left[\frac{\|u\|^2}{2} + \lambda(\rho_0 + \Phi_1(u))\right]$$

is obviously upper semicontinuous on $\mathbb{I}$. It follows from (3.7) that

$$\lim_{\lambda\to+\infty}\inf_{u\in W^{1,2}(\Omega)}\varphi(u,\lambda)\le\lim_{\lambda\to+\infty}\left[\frac{\|u_0\|^2}{2}+\lambda(\rho_0+\Phi_1(u_0))\right]=-\infty.$$

Thus we find an element $\overline{\lambda}\in\mathbb{I}$ such that

$$\sup_{\lambda\in\mathbb{I}}\inf_{u\in W^{1,2}(\Omega)}\varphi(u,\lambda)=\inf_{u\in W^{1,2}(\Omega)}\left[\frac{\|u\|^2}{2}+\overline{\lambda}(\rho_0+\Phi_1(u))\right].\tag{3.9}$$

Since $-\beta(t_0)<\rho_0$, it follows from the definition of $\beta$ that for all $u\in W^{1,2}(\Omega)$ with $\frac{\|u\|^2}{2}<t_0$ we have $-\Phi_1(u)<\rho_0$. Hence

$$t_0\le\inf\left\{\frac{\|u\|^2}{2}:u\in W^{1,2}(\Omega),\ -\Phi_1(u)\ge\rho_0\right\}.\tag{3.10}$$

On the other hand,

$$\begin{aligned}\inf_{u\in W^{1,2}(\Omega)}\sup_{\lambda\in\mathbb{I}}\varphi(u,\lambda)&=\inf_{u\in W^{1,2}(\Omega)}\left[\frac{\|u\|^2}{2}+\sup_{\lambda\in\mathbb{I}}(\lambda(\rho_0+\Phi_1(u)))\right]\\&=\inf_{u\in W^{1,2}(\Omega)}\left\{\frac{\|u\|^2}{2}:-\Phi_1(u)\ge\rho_0\right\}.\end{aligned}$$

Thus inequality (3.10) is equivalent to

$$t_0\le\inf_{u\in W^{1,2}(\Omega)}\sup_{\lambda\in\mathbb{I}}\varphi(u,\lambda).\tag{3.11}$$

We consider two cases. First, when $0\le\overline{\lambda}<\frac{t_0}{\rho_0}$, then we have that

$$\inf_{u\in W^{1,2}(\Omega)}\left[\frac{\|u\|^2}{2}+\overline{\lambda}(\rho_0+\Phi_1(u))\right]\le\varphi(0,\overline{\lambda})=\overline{\lambda}\rho_0<t_0.$$

Combining this inequality with (3.9) and (3.11) we obtain (3.8).

Now, if $\frac{t_0}{\rho_0}\le\overline{\lambda}$, then from (3.6) and (3.7), it follows that

$$\begin{aligned}\inf_{u\in W^{1,2}(\Omega)}\left[\frac{\|u\|^2}{2}+\overline{\lambda}(\rho_0+\Phi_1(u))\right]&\le\frac{\|u_0\|^2}{2}+\overline{\lambda}(\rho_0+\Phi_1(u_0))\\&\le\frac{\|u_0\|^2}{2}+\frac{t_0}{\rho_0}(\rho_0+\Phi_1(u_0))<t_0.\end{aligned}$$

It remains to apply again (3.9) and (3.11), which concludes the proof of (3.8).

Now, we are in the position to apply Theorem 2.1; we choose $X=W^{1,2}(\Omega)$, $\tilde{X}_1=L^p(\Omega)$ with $p\in[1,2^*[$, $\tilde{X}_2=L^q(\partial\Omega)$ with $q\in[1,\overline{2}^*[$, $\Lambda=\mathbb{I}=[0,+\infty)$, $h(t)=t^2/2$, $t\ge0$.

Now, we fix $g\in\mathcal{G}_\tau$ $(\tau\ge0)$, $\lambda\in\Lambda$, $\mu\in[0,\lambda+1]$, and $c\in\mathbb{R}$. We shall prove that the functional $E_{\lambda,\mu}:W^{1,2}(\Omega)\to\mathbb{R}$ given by

$$E_{\lambda,\mu}(u)=\frac{1}{2}\|u\|^2+\lambda\Phi_1(u)+\mu(g\circ\Phi_2)(u),\quad u\in W^{1,2}(\Omega),$$

satisfies the $(PS)_c$. Note that due to Lemma 1.1, we have for every $u,v\in W^{1,2}(\Omega)$ that

$$E^\circ_{\lambda,\mu}(u;v)\le\langle u,v\rangle_{W^{1,2}}+\lambda\Phi_1^\circ(u;v)+\mu(g\circ\Phi_2)^\circ(u;v).\tag{3.12}$$

First of all, let us observe that $\frac{1}{2}\|\cdot\|^2 + \lambda\Phi_1$ is coercive on $W^{1,2}(\Omega)$, due to **(F2)**; thus, the functional $E_{\lambda,\mu}$ is also coercive on $W^{1,2}(\Omega)$. Consequently, it is enough to consider a bounded sequence $\{u_n\} \subset W^{1,2}(\Omega)$ such that

$$E^{\circ}_{\lambda,\mu}(u_n; v - u_n) \geq -\varepsilon_n \|v - u_n\| \quad \text{for all } v \in W^{1,2}(\Omega), \tag{3.13}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n \to 0$. Because the sequence $\{u_n\}$ is bounded, there exists an element $u \in W^{1,2}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$, $u_n \to u$ strongly in $L^p(\Omega)$, $p \in [1, 2^*[$ (since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is compact), and $u_n \to u$ strongly in $L^q(\partial\Omega)$, $q \in [1, \overline{2}^*[$ (since $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact). Using (3.13) with $v = u$ and apply relation (3.12) for the pairs $(u_n, u - u_n)$ and $(u, u_n - u)$, we have that

$$\|u - u_n\|^2 \leq \varepsilon_n \|u - u_n\| - E^{\circ}_{\lambda,\mu}(u; u_n - u) + \lambda[\Phi^{\circ}_1(u_n; u - u_n) + \Phi^{\circ}_1(u; u_n - u)] \\ +\mu[(g \circ \Phi_2)^{\circ}(u_n; u - u_n) + (g \circ \Phi_2)^{\circ}(u; u_n - u)].$$

Since $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$, we clearly have that $\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0$. Now, fix $z^* \in \partial E^{\circ}_{\lambda,\mu}(u)$; in particular, we have $\langle z^*, u_n - u\rangle_{W^{1,2}} \leq E^{\circ}_{\lambda,\mu}(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$, we have that $\liminf_{n\to\infty} E^{\circ}_{\lambda,\mu}(u; u_n - u) \geq 0$. Now, for the remaining four terms in the above estimation we use the fact that $\Phi^{\circ}_1(\cdot;\cdot)$ and $(g \circ \Phi_2)^{\circ}(\cdot;\cdot)$ are upper semicontinuous functions on $L^p(\Omega)$ and $L^q(\partial\Omega)$, respectively. Since $u_n \to u$ strongly in $L^p(\Omega)$, we have for instance $\limsup_{n\to\infty} \Phi^{\circ}_1(u_n; u - u_n) \leq \Phi^{\circ}_1(u; 0) = 0$; the remaining terms are similar. Combining the above outcomes, we obtain finally that $\limsup_{n\to\infty} \|u - u_n\|^2 \leq 0$, i.e., $u_n \to u$ strongly in $W^{1,2}(\Omega)$. It remains to apply Theorem 2.1 in order to obtain the conclusion. □

*Remark 3.1* Marano and Papageorgiou [11] studied a similar problem to $(P_{\lambda,\mu})$ by considering the homogeneous case when $G = 0$ and the $p$-Laplacian operator $\triangle_p$ instead of the standard Laplacian $\triangle$. By using a non-smooth mountain pass type argument (with zero altitude), they guaranteed the existence of solutions for the studied problem.

### 3.2 A differential inclusion in $\mathbb{R}^N$

Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

($\tilde{\mathbf{F}}$1) $\displaystyle\lim_{t\to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{|t|^{p-1}} = 0;$

($\tilde{\mathbf{F}}$2) $\displaystyle\limsup_{|t|\to+\infty} \frac{F(t)}{|t|^p} \leq 0;$

($\tilde{\mathbf{F}}$3) There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

In this section we are going to study the differential inclusion problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) & \text{on } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $p > N \geq 2$, the numbers $\lambda, \mu$ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^N)$ is any function, and
($\tilde{\alpha}$) $\alpha \in L^1(\mathbb{R}^N) \cap L^{\infty}_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 0$, and $\sup_{R>0} \operatorname{essinf}_{|x|\leq R} \alpha(x) > 0$.

The functional space where our solutions are going to be sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with the norm $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + \int_{\mathbb{R}^N} |u(x)|^p\right)^{1/p}$.

**Definition 3.2** We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a solution of problem $(\tilde{P}_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for a. e. $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)\mathrm{d}x = \lambda \int_{\mathbb{R}^N} \alpha(x)\xi_F v\mathrm{d}x + \mu \int_{\mathbb{R}^N} \beta(x)\xi_G v\mathrm{d}x. \quad (3.14)$$

*Remark 3.2* (a) The terms in the right hand side of (3.14) are well-defined. Indeed, due to Morrey's embedding theorem, i.e., $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous $(p > N)$, we have $u \in L^\infty(\mathbb{R}^N)$. Thus, there exists a compact interval $I_u \subset \mathbb{R}$ such that $u(x) \in I_u$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping $\partial F$ is upper-semicontinuous, the set $\partial F(I_u) \subset \mathbb{R}$ is bounded; let $C_F = \sup|\partial F(I_u)|$. Therefore,

$$\left|\int_{\mathbb{R}^N} \alpha(x)\xi_F v\mathrm{d}x\right| \leq C_F\|\alpha\|_{L^1}\|v\|_\infty < \infty.$$

Similar argument holds for the function $G$.

(b) Since $p > N$, any element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$, see Brézis [5, Théorème IX.12].

The main result of this section is

**Theorem 3.2** *Assume that $p > N \geq 2$. Let $\alpha, \beta \in L^1(\mathbb{R}^N)$ be two radial functions, $\alpha$ fulfilling $(\tilde{\alpha})$, and let $F, G : \mathbb{R} \to \mathbb{R}$ be two locally Lipschitz functions, $F$ satisfying the conditions* ($\tilde{\mathbf{F}}$1–$\tilde{\mathbf{F}}$3).*Then there exists a non-degenerate compact interval $[a, b] \subset ]0, +\infty[$ and a number $\tilde{r} > 0$, such that for every $\lambda \in [a, b]$ there exists $\mu_0 \in ]0, \lambda + 1]$ such that for each $\mu \in [0, \mu_0]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with $L^\infty$-norms less than $\tilde{r}$.*

Note that no hypothesis on the growth of $G$ is assumed; therefore, the last term in $(\tilde{P}_{\lambda,\mu})$ may have an arbitrary growth. However, assumption $(\tilde{\alpha})$ together with ($\tilde{\mathbf{F}}$3) guarantee the existence of non-trivial solutions for $(\tilde{P}_{\lambda,\mu})$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1; we will show only the differences. To do that, we introduce some notions and preliminary results.

Although the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous (due to Morrey's theorem $(p > N)$), it is not compact. We overcome this gap by introducing the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$. The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) = u(g^{-1}x)$, for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this group acts linearly and isometrically; in particular $\|gu\| = \|u\|$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Defining the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ by

$$W^{1,p}_{\text{rad}}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \quad \text{for all } g \in O(N)\},$$

we can state the following result.

**Proposition 3.2** [9] *The embedding $W^{1,p}_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact whenever $2 \leq N < p < \infty$.*

Let $\Phi_1, \Phi_2 : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))\mathrm{d}x \quad \text{and} \quad \Phi_2(u) = -\int_{\mathbb{R}^N} \beta(x)G(u(x))\mathrm{d}x.$$

Since $\alpha, \beta \in L^1(\mathbb{R}^N)$, the functionals $\Phi_1$, $\Phi_2$ are well-defined and locally Lipschitz, see Clarke [7, p. 79-81]. Moreover, we have

$$\partial \Phi_1(u) \subseteq -\int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))\mathrm{d}x, \quad \partial \Phi_2(u) \subseteq -\int_{\mathbb{R}^N} \beta(x)\partial G(u(x))\mathrm{d}x.$$

The energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\tilde{P}_{\lambda,\mu})$, is given by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu\Phi_2(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

It is clear that the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(\tilde{P}_{\lambda,\mu})$ in the sense of Definition 3.2; for a similar argument, see Kristály [9].

Since $\alpha, \beta$ are radially symmetric, then $\mathcal{E}_{\lambda,\mu}$ is $O(N)$-invariant, i.e. $\mathcal{E}_{\lambda,\mu}(gu) = \mathcal{E}_{\lambda,\mu}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we may apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz-Marzantowicz [8], whose form in our setting is as follows.

**Proposition 3.3** *Any critical point of* $\mathcal{E}^{\mathrm{rad}}_{\lambda,\mu} = \mathcal{E}_{\lambda,\mu}|_{W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)}$ *will be also a critical point of* $\mathcal{E}_{\lambda,\mu}$.

The following result can be compared with Proposition 3.1, although their proofs are different.

**Proposition 3.4** $\lim_{t\to 0^+} \dfrac{\inf\{\Phi_1(u) :\ u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N),\ \|u\|^p < pt\}}{t} = 0.$

*Proof* Due to ($\tilde{\mathbf{F}}$1), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\xi| \le \varepsilon |t|^{p-1}, \quad \forall t \in [-\delta(\varepsilon), \delta(\varepsilon)],\ \forall \xi \in \partial F(t). \tag{3.15}$$

For any $0 < t \le \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ define the set

$$S_t = \{\, u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) : \|u\|^p < pt\},$$

where $c_\infty > 0$ denotes the best constant in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Note that $u \in S_t$ implies that $\|u\|_\infty \le \delta(\varepsilon)$; indeed, we have $\|u\|_\infty \le c_\infty\|u\| < c_\infty(pt)^{1/p} \le \delta(\varepsilon)$. Fix $u \in S_t$; for a.e. $x \in \mathbb{R}^N$, Lebourg's mean value theorem and (3.15) imply the existence of $\xi_x \in \partial F(\theta_x u(x))$ for some $0 < \theta_x < 1$ such that

$$F(u(x)) = F(u(x)) - F(0) = \xi_x u(x) \le |\xi_x| \cdot |u(x)| \le \varepsilon |u(x)|^p.$$

Consequently, for every $u \in S_t$ we have

$$\begin{aligned}\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))\mathrm{d}x &\ge -\varepsilon\int_{\mathbb{R}^N} \alpha(x)|u(x)|^p\mathrm{d}x\\ &\ge -\varepsilon\|\alpha\|_{L^1}\|u\|_\infty^p \ge -\varepsilon\|\alpha\|_{L^1}c_\infty^p\|u\|^p\\ &\ge -\varepsilon\|\alpha\|_{L^1}c_\infty^p pt.\end{aligned}$$

Therefore, for every $0 < t \le \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ we have

$$0 \ge \frac{\inf_{u\in S_t}\Phi_1(u)}{t} \ge -\varepsilon\|\alpha\|_{L^1}c_\infty^p p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required limit. □

*Proof of Theorem 3.2* We are going to apply Theorem 2.1 by choosing $X = W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, $\tilde{X}_1 = \tilde{X}_2 = L^\infty(\mathbb{R}^N)$, $\Lambda = [0, +\infty)$, $h(t) = t^p/p$, $t \geq 0$.

Fix $g \in \mathcal{G}_\tau$ ($\tau \geq 0$), $\lambda \in \Lambda$, $\mu \in [0, \lambda + 1]$, and $c \in \mathbb{R}$. We prove that the functional $E_{\lambda,\mu} : W^{1,p}_{\text{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu(g \circ \Phi_2)(u), \quad u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N),$$

satisfies the $(PS)_c$ condition.

Note first that the function $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$ is coercive on $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. To prove this, let $0 < \varepsilon < (p\|\alpha\|_1 c_\infty^p \lambda)^{-1}$. Then, on account of ($\tilde{\mathbf{F}}$2), there exists $\delta(\varepsilon) > 0$ such that

$$F(t) \leq \varepsilon|t|^p, \quad \forall |t| > \delta(\varepsilon).$$

Consequently, for every $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ we have

$$\begin{aligned}
\Phi_1(u) &= -\int_{\mathbb{R}^N} \alpha(x)F(u(x))\mathrm{d}x \\
&= -\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta(\varepsilon)\}} \alpha(x)F(u(x))\mathrm{d}x - \int_{\{x\in\mathbb{R}^N:|u(x)|\leq\delta(\varepsilon)\}} \alpha(x)F(u(x))\mathrm{d}x \\
&\geq -\varepsilon\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta(\varepsilon)\}} \alpha(x)|u(x)|^p\mathrm{d}x - \max_{|t|\leq\delta(\varepsilon)}|F(t)|\int_{\{x\in\mathbb{R}^N:|u(x)|\leq\delta(\varepsilon)\}} \alpha(x)\mathrm{d}x \\
&\geq -\varepsilon\|\alpha\|_{L^1} c_\infty^p\|u\|^p - \|\alpha\|_{L^1}\max_{|t|\leq\delta(\varepsilon)}|F(t)|.
\end{aligned}$$

Now, we have

$$\frac{1}{p}\|u\|^p + \lambda\Phi_1(u) \geq \left(\frac{1}{p} - \varepsilon\lambda\|\alpha\|_{L^1}c_\infty^p\right)\|u\|^p - \lambda\|\alpha\|_{L^1}\max_{|t|\leq\delta(\varepsilon)}|F(t)|,$$

which clearly implies the coercivity of $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$.

As an immediate consequence, the functional $E_{\lambda,\mu}$ is also coercive on $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Therefore, it is enough to consider a bounded sequence $\{u_n\} \subset W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that

$$E^\circ_{\lambda,\mu}(u_n; v - u_n) \geq -\varepsilon_n\|v - u_n\| \quad \text{for all} \;\; v \in W^{1,p}_{\text{rad}}(\mathbb{R}^N), \tag{3.16}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n \to 0$. Since the sequence $\{u_n\}$ is bounded in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, one can find an element $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, and $u_n \to u$ strongly in $L^\infty(\mathbb{R}^N)$, due to Proposition 3.2.

Due to Lemma 1.1, for every $u, v \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ we have

$$E^\circ_{\lambda,\mu}(u; v) \leq \int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv) + \lambda\Phi_1^\circ(u; v) + \mu(g \circ \Phi_2)^\circ(u; v). \tag{3.17}$$

Put $v = u$ in (3.16) and apply relation (3.17) for the pairs $(u, v) = (u_n, u - u_n)$ and $(u, v) = (u, u_n - u)$, we have that

$$\begin{aligned}
I_n \leq\; &\varepsilon_n\|u - u_n\| - E^\circ_{\lambda,\mu}(u; u_n - u) + \lambda[\Phi_1^\circ(u_n; u - u_n) + \Phi_1^\circ(u; u_n - u)] \\
&+\mu[(g \circ \Phi_2)^\circ(u_n; u - u_n) + (g \circ \Phi_2)^\circ(u; u_n - u)],
\end{aligned}$$

where

$$I_n \stackrel{\text{not.}}{=} \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u) + \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u).$$

Since $\{u_n\}$ is bounded in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0$. Fixing $z^* \in \partial E^{\circ}_{\lambda,\mu}(u)$ arbitrarily, we have $\langle z^*, u_n - u\rangle \leq E^{\circ}_{\lambda,\mu}(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\liminf_{n\to\infty} E^{\circ}_{\lambda,\mu}(u; u_n - u) \geq 0$. The functions $\Phi_1^{\circ}(\cdot;\cdot)$ and $(g \circ \Phi_2)^{\circ}(\cdot;\cdot)$ are upper semicontinuous functions on $L^{\infty}(\mathbb{R}^N)$. Since $u_n \to u$ strongly in $L^{\infty}(\mathbb{R}^N)$, the upper limit of the last four terms is less or equal than 0 as $n \to \infty$, see Lemma 1.1 d).

Consequently,

$$\limsup_{n\to\infty} I_n \leq 0. \tag{3.18}$$

Since $|t - s|^p \leq (|t|^{p-2}t - |s|^{p-2}s)(t - s)$ for every $t, s \in \mathbb{R}^m$ $(m \in \mathbb{N})$ we infer that $\|u_n - u\|^p \leq I_n$. The last inequality combined with (3.18) leads to the fact that $u_n \to u$ strongly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, as claimed.

It remains to prove relation (2.1) from Theorem 2.1. On account of Proposition 3.4, this part can be completes in a similar way as we performed in the proof of Theorem 3.1, the only difference is the construction of the function $u_0$ appearing after relation (3.5). In the sequel, we construct the corresponding function $u_0 \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that $\Phi_1(u_0) < 0$.

On account of $(\tilde{\alpha})$, one can fix $R > 0$ such that $\alpha_R = \operatorname{essinf}_{|x|\leq R}\alpha(x) > 0$. For $\sigma \in ]0, 1[$ define the function

$$w_\sigma(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_N(0, R); \\ \tilde{t}, & \text{if } x \in B_N(0, \sigma R); \\ \frac{\tilde{t}}{R(1-\sigma)}(R - |x|), & \text{if } x \in B_N(0, R) \setminus B_N(0, \sigma R), \end{cases}$$

where $B_N(0, r)$ denotes the $N$-dimensional open ball with center 0 and radius $r > 0$, and $\tilde{t}$ comes from $(\tilde{\mathbf{F}}3)$. Since $\alpha \in L^{\infty}_{\text{loc}}(\mathbb{R}^N)$, then $M(\alpha, R) = \sup_{x\in B_N(0,R)} \alpha(x) < \infty$. A simple estimate shows that

$$-\Phi_1(w_\sigma) \geq \omega_N R^N \left[\alpha_R F(\tilde{t})\sigma^N - M(\alpha, R) \max_{|t|\leq|\tilde{t}|} |F(t)|(1 - \sigma^N)\right].$$

When $\sigma \to 1$, the right hand side is strictly positive; choosing $\sigma_0$ close enough to 1, for $u_0 = w_{\sigma_0}$ we have $\Phi_1(u_0) < 0$.

Due to Theorem 2.1, there exist a non-empty open set $A \subset \Lambda$ and $r > 0$ with the property that for every $\lambda \in A$ there exists $\mu_0 \in ]0, \lambda + 1]$ such that, for each $\mu \in [0, \mu_0]$ the functional $\mathcal{E}^{\text{rad}}_{\lambda,\mu} = \frac{1}{p}\|\cdot\|^p + \lambda\Phi_1 + \mu\Phi_2$ defined on $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ has at least three critical points in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ whose $\|\cdot\|$-norms are less than $r$. Applying Proposition 3.3, the critical points of $\mathcal{E}^{\text{rad}}_{\lambda,\mu}$ are also critical points of $\mathcal{E}_{\lambda,\mu}$, thus, radially weak solutions of problem $(\tilde{P}_{\lambda,\mu})$. Due to the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\infty}(\mathbb{R}^N)$, if $\tilde{r} = c_\infty r$, then the $L^{\infty}$-norms of these elements are less than $\tilde{r}$ which concludes our proof. □

**Acknowledgment** We would like to thank the anonymous Referees for their useful remarks and comments. A. Kristály and Cs. Varga are supported by Grant PN II, ID_2162 and by Project PNCDI II/Idei/2008/C_Exploratorie no. 55 from CNCSIS.

## References

1. Arcoya, D., Carmona, J.: A nondifferentiable extension of a theorem of Pucci-Serrin and applications. J. Differ. Equ. **235**(2), 683–700 (2007)
2. Bonanno, G.: Some remarks on a three critical points theorem. Nonlinear Anal. **54**, 651–665 (2003)
3. Bonanno, G.: A critical points theorem and nonlinear differential problems. J. Global Optim. **28**(3–4), 249–258 (2004)
4. Bonanno, G., Candito, P.: Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearity. J. Differ. Equ. **244**(12), 3031–3059 (2008)
5. Brézis, H.: Analyse Fonctionnelle-Théorie et Applications. Masson, Paris (1992)
6. Chang, K.C.: Variational methods for non-differentiable functionals and their applications to partial differential equations .J. Math. Anal. Appl. **80**, 102–129 (1981)
7. Clarke, F.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
8. Krawcewicz, W., Marzantowicz, W.: Some remarks on the Lusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action. Rocky Mt. J. Math. **20**, 1041–1049 (1990)
9. Kristály, A.: Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$. J. Differ. Equ. **220**, 511–530 (2006)
10. Marano, S.A., Motreanu, D.: On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems. Nonlinear Anal. **48**, 37–52 (2002)
11. Marano, S.A., Papageorgiou, N.S.: On a Neumann problem with $p$-Laplacian and non-smooth potential. Differ. Integral Equ. **19**(11), 1301–1320 (2006)
12. Motreanu, D., Varga, Cs.: Some critical point results for locally Lipschitz functionals. Comm. Appl. Nonlinear Anal. **4**, 17–33 (1997)
13. Pucci, P., Serrin, J.: A mountain pass theorem. J. Differ. Equ. **60**, 142–149 (1985)
14. Ricceri, B.: On a three critical points theorem. Arch. Math. (Basel) **75**, 220–226 (2000)
15. Ricceri, B.: Existence of three solutions for a class of elliptic eigenvalue problems. Math. Comput. Model. **32**, 1485–1494 (2000)
16. Ricceri, B.: Three solutions for a Neumann problem. Topol. Methods Nonlinear Anal. **20**, 275–282 (2002)
17. Ricceri, B.: A three critical points theorem revisited. Nonlinear Anal. doi:10.1016/j.na.2008.04.010.
18. Ricceri, B.: Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set. Topol. Appl. **153**, 3308–3312 (2006)